A NOTE ON THE BIRATIONAL GEOMETRY OF TROPICAL LINE BUNDLES

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Abstract. Given a closed subvariety Y of a n-dimensional torus, we study how the tropical line bundles of $\mathrm{Trop}(Y)$ can be induced by line bundles living on a tropical compactification of Y in a toric variety, following the construction of J. Tevelev. We then consider the general structure with respect to the Zariski–Riemann space.

Introduction

The aim of this work is to give a birational interpretation to the systematic study of line bundles on tropical varieties (see for example [AR10]). In particular we are interested in understanding how much of the structure of tropical line bundles can be reconstructed from the line bundles living in the original variety.

We will always consider a closed subvariety Y of a complex torus T. The main result is that tropical compactifications in the sense of [Tev07] control the possible ways of a line bundle on Y to extend to a compactification $\overline{Y}$ of Y induced by the tropicalization in some smooth toric variety. As expected, this choice is not unique. It will infact correspond to a toric b-divisor.

The machinery is quite complicated, but we will try to give as much as possible a self-contained work, emphasizing the natural connection between the way tropical geometry induces compactifications of subvarieties of tori and the birational geometry of the space where the variety gets compactified.

This approach can be thought as a higher dimensional generalization of Matthew Baker's work [Bak08] for curves and a more explicit connection to the Zariski–Riemann space as the one given in [HU14].

In Sections 1 and 2 we will first introduce toric and tropical varieties, the main objects of our study. Then we will describe how the tropicalization of Y induces a (tropical) compactification in some toric variety in the sense of [Tev07]. This compactification will have all sorts of good properties that we will consistently use throughout the rest of the work.

In Section 3 we will introduce the Minkowski weights and we will describe how a tropical compactification of Y induces a Minkowski weight. Minkowski weights are the objects that correspond to line bundles in the language of tropical geometry. With this in mind we will start our construction.

In Section 4 we will see how the compactification $\overline{Y} \hookrightarrow \mathbb{P}_{\mathrm{Trop}(Y)}$ in the toric variety $\mathbb{P}_{\mathrm{Trop}(Y)}$ induces a toric divisor in $\mathbb{P}_{\mathrm{Trop}(Y)}$ coming from a line bundle on $\overline{Y}$, where $\overline{Y}$ is a tropical compactification of a very affine variety Y. We will describe

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some properties of these bundles and how they behave from a birational point of view. It is important to notice that the compactification is not unique, hence it is natural to put together all this compactifications and the induced bundles. This will be done in section 4.1, identifying these bundles with an open subspace of the toric Zariski–Riemann space associated with any compactification $\mathbb{P}_{\mathrm{Trop}(Y)}$.

In Section 5 we finally give examples of the explicit construction and we will show how the compactification induces a toric structure of the Picard group of the variety we are compactifying.

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1. Toric, tropical and very affine varieties

1.1. **Toric varieties.** Let us recall the basic definitions and properties of toric varieties from [CLS11] that will be important for our construction.

Let $\mathbb{P}_\Sigma$ be the normal toric variety corresponding to the fan Σ in $N_\mathbb{R} \cong \mathbb{R}^n$, T_N being the associated maximal torus. Recall that every T_N-invariant Weil divisor is represented by a sum

$$D = \sum_{\rho \in \Sigma(1)} d_\rho D_\rho,$$

where ρ is a one-dimensional subcone (a ray) of Σ, and D_ρ is the associated T_N-invariant prime divisor. Such a divisor D is Cartier if and only if, for every maximal dimension subcone $\sigma \in \Sigma(n)$, $D|_{U_\sigma}$ is locally a divisor of a character $\mathrm{div}(\chi^{m_\sigma})$, with $m_\sigma \in N^\vee = M$. If D is Cartier we will say that $\{m_\sigma | \sigma \in \Sigma(n)\}$ is the Cartier data of D.

To every divisor we can associate a polyhedron:

$$P_D = \{m \in M_\mathbb{R} | \langle m, u_\rho \rangle \geq -d_\rho \text{ for every } \rho \in \Sigma(1)\},$$

where u_ρ is the minimal generator of the ray ρ.

For every divisor D and every cone $\sigma \in \Sigma(n)$, we can describe the local sections as

$$\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[W]$$

where $W = \{\chi^m | \langle m, u_\rho \rangle + d_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\}$.

Proposition 1.1.1. *Let D be a Cartier divisor on a toric variety $\mathbb{P}_\Sigma$ whose fan has convex support of full dimension. Then D is linearly equivalent to* 0 *if and only if it is numerically equivalent to* 0.

Remark 1.1.2. This in particular implies that it is possible to identify a line bundle via its intersections with T-invariant curves. More in general we would like to identify toric bundles via intersections with T-invariant subschemes by induction on the dimension.

Remark 1.1.3. Most times the fan giving the toric varieties we are constructing will not satisfy the condition of being full-dimensional. This because the statement will be independent of the chosen compactification.

1.2. **Very affine varieties.** We start introducing the notion of very affine varieties, that will be the crucial objects in the rest of the paper.

Definition 1.2.1. A *very affine variety* is a closed subvariety of an algebraic torus.

Notice that this definition involves the choice of an algebraic torus as an ambient space. The following definition gives a canonical choice.

Definition 1.2.2. Let $Y \subseteq T_Y$ be a very affine variety. The torus T_Y with character lattice $M_Y := \mathcal{O}^*(Y)/\mathbb{C}^*$, where the action of $\mathbb{C}^*$ is a diagonal action on a choice of generators of $\mathcal{O}^*(Y)$, is called *the universal torus of* Y (by Iitaka in [Iit82]) or *the intrinsic torus of* Y (by Tevelev in [Tev07, Section 3]).

Proposition 1.2.3. *If Y is a very affine subvariety, then Y is a closed subvariety of the torus T_Y. Moreover, the embedding is universal in the sense that any map $Y \to T$ from Y to an algebraic torus T factors as $Y \to T_Y \to T$, where the second map is a homomorphism of tori.*

Remark 1.2.4. Notice that the character lattice M_Y is a finite rank lattice and the embedding of Y in T_Y is given by the evaluation map

$$p \mapsto (f \mapsto f(p)).$$

Corollary 1.2.5. *Y is very affine if and only if it is a closed subvariety of T_Y.*

1.3. **Tropicalization of very affine varieties.** Let K be a field with a non-Archimedean discrete valuation

$$val\colon \overline{K}^* \to \mathbb{Q}.$$

Let $\mathcal{O}$ be its valuation ring with residue field $\mathbf{k}$.

For any fixed natural number n, let $\mathcal{T} \cong \mathbb{G}_m^n$ be the split n-dimensional torus over $\mathcal{O}$ and $T = \mathcal{T} \times_{\mathcal{O}} \mathbb{K}$ the corresponding torus over $\mathbb{K}$. Let

$$Val\colon T \to \mathbb{Q}^n$$

be the valuation induced by val.

Definition 1.3.1. Let Y is a very affine connected subvariety of a torus T. The *tropical variety* $\operatorname{Trop}(Y)$ *associated with* Y is the closure $\overline{Val(Y)}$ of $Val(Y)$ in $\mathbb{R}^n$.

Remark 1.3.2. There are several equivalent definitions of tropicalization. Here we are thinking of it as the non-Archimedean amoeba associated with the given valuation.

2. Tropical compactifications of algebraic varieties

2.1. **Tropical fan and tropical compactification.** By [BG84] we know that tropical varieties have the structure of polyhedral complexes and it is well-known that the tropical variety associated with Y respects the properties of an abstract tropical variety.

Definition 2.1.1. [Tev07, Definitions 1.1, 1.3 and 2.4] Let Y be a very affine and irreducible proper subvariety of a torus T and let $\overline{Y}$ be the Zariski closure of Y in a toric variety $\mathbb{P}_\Sigma$ containing T. Let $\overline{m}\colon \overline{Y} \times T \to \mathbb{P}_\Sigma$ be the multiplication map.

The fan Σ is a *tropical fan* for Y if $\overline{m}$ is flat and surjective. In this case we say that $(Y, \mathbb{P}_\Sigma)$ is a *tropical pair* and that $\overline{Y}$ is *a tropical compactification of* Y.

The fan Σ is a *schön fan* for Y if $\overline{m}$ is smooth and surjective, *i.e.* Y has a tropical compactification with a smooth multiplication map. In this case we say that Y is *schön.*

Remark 2.1.2. Any schön fan is trivially a tropical fan.

The key point is that the tropicalization of a very affine variety carries properties of the underlying toroidal structure. Let us start recalling one of the main statements in this direction:

Proposition 2.1.3. [Tev07, Proposition 2.5] *Let Y a very affine and connected variety and let Σ be a tropical fan for Y. Then $|\Sigma| = \mathrm{Trop}(Y)$.*
Moreover, if Σ' is any refinement of Σ, then also Σ' is tropical for Y.

The last statement allows one to frequently assume without loss of generality that a tropical fan is smooth.

Remark 2.1.4. If Y is a very affine and connected variety, by the main result in [Tev07] there exists a fan which is tropical for Y (see Theorem 2.3.1).

Remark 2.1.5. Recall that $\mathrm{Trop}(Y)$ does not necessarily come with a preferred fan structure and that not every fan structure on $\mathrm{Trop}(Y)$ is tropical.

We will now give an explicit procedure for constructing tropical fans.

2.2. **Toric fan induced by the tropicalization.**

The following constructioon comes from [KS12, Section 2]. Let Σ be a subcomplex of a rational polyhedral subdivision of $\mathbb{R}^n$ supported on $\mathrm{Trop}(X)$. We will define a toric scheme $\mathbb{P}_\Sigma$ over $\mathcal{O}$.

- let us first consider $\Sigma \times \{1\} \subseteq N_\mathbb{R} \times \mathbb{R}_{\geq 0}$;
- let $\tilde{\Sigma}$ be the fan of cones supported on Σ in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$, i.e. the set of cones over the facets of $\Sigma \times \{1\}$;
- note that $\tilde{\Sigma}$ is not complete and let $\mathbb{P}_{\tilde{\Sigma}}$ the corresponding toric variety;
- let $\mathbb{P}_\Sigma = \mathbb{P}_{\tilde{\Sigma}} \times_{\mathbb{Z}[u]} \mathcal{O}$ where $\mathrm{Spec}\mathcal{O} \to \mathrm{Spec}\mathbb{Z}[u]$ is the morphism induced by the projection $\mathbb{P}_{\tilde{\Sigma}} \to \mathbb{A}^1$ (induced by $N_\mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$);
- denoting by Σ the recession fan of Σ, then we have that the toric variety $\mathbb{P}_\Sigma$ is the generic fiber of $\mathbb{P}_{\tilde{\Sigma}}$.

Remark 2.2.1. It is fundamental to notice that if Val is the trivial valuation, the set Σ corresponds to the one denoted *Σ_0* in [BG84] that without any additional construction is the support of a polyhedral fan. In this case then $\mathbb{P}_{\tilde{\Sigma}} \cong \mathbb{P}_\Sigma$ can be compactified in a toric variety $\overline{\mathbb{P}_\Sigma}$ such that $\overline{\mathbb{P}_\Sigma} \backslash \mathbb{P}_\Sigma$ has codimension ≥ 2.

Remark 2.2.2. $\mathbb{P}_\Sigma$ is smooth if and only if Σ is unimodular, *i.e.* each cone of Σ is generated by a subset of an integral basis of N. Recall that each fan can be refined to be unimodular, inducing a resolution of singularities via subdivisions of cones.

2.3. **Main properties of tropical compactifications.**

Theorem 2.3.1. [Tev07, Theorem 1.2] *Let Y be a very affine and connected variety. Then Y has a tropical compactification $\overline{Y}$ in a smooth toric variety $\mathbb{P}_\Sigma$. Moreover, the boundary $\overline{Y} \setminus Y$ is a reduced divisor on $\overline{Y}$; if $D_1, \dots, D_k$ are distinct irreducible components of $\overline{Y} \setminus Y$, then $D_1 \cap \cdots \cap D_k$ is either empty or of pure codimension k.*

Notice that the last condition in the theorem above can be reformulated as

$$\dim(\operatorname{orb}(\sigma) \cap \overline{Y}) = \dim Y - \dim \sigma \text{ for any cone } \sigma \in \Sigma.$$

Proposition 2.3.2. *Let Y be a very affine and irreducible variety of dimension k. Then* $\operatorname{Trop}(Y)$ *has real dimension k as a polyhedral complex.*

Proposition 2.3.3. [Tev07, Proposition 2.3] *Let Y be a very affine and connected variety with a tropical compactification $\overline{Y}$ in a smooth toric variety $\mathbb{P}_\Sigma$ Then $\overline{Y}$ intersects torus orbits properly if and only if* $\operatorname{Trop}(Y)$ *is a union of cones of* Σ.

Theorem 2.3.4. [Tev07, Theorem 2.5] *If Y is smooth and admits a schön fan, then, for any fan Σ which is tropical for X, the corresponding compactification $\overline{Y}$ has toroidal singularities.*

Proposition 2.3.5. [LQ11] *If Y has a schön fan, then any smooth fan Σ supported on* $\operatorname{Trop}(Y)$ *is also schön for Y.*

Theorem 2.3.6. [Tev07, Theorem 1.4] *Let Y to be a very affine and connected subvariety of the torus T. If Y is schön for some tropical pair $(Y, \mathbb{P}_\Sigma)$, then it is schön for any tropical pair.*

Remark 2.3.7. [Kat12, p. 34] Let Y to be a very affine and connected subvariety of the torus T. If $(Y, \mathbb{P}_\Sigma)$ is schön and $\mathbb{P}_\Sigma$ is smooth, then the stratification of $\mathbb{P}_\Sigma$ as a toric variety pulls-back to Y as a stratification with smooth strata.

3. Minkowski weights

3.1. **Definition.** Let $\mathbb{P}_\Sigma$ be a n-dimensional normal toric variety with torus T.

Recall that $\mathbb{P}_\Sigma$ is determined by the rational fan Σ in a vector space $N_\mathbb{R}$ and that T has a one-parameter subgroup lattice canonically isomorphic to a lattice $N \subset N_\mathbb{R}$. If τ is a cone in Σ, we denote by Σ_τ a fan whose cones are $\overline{\sigma} := (\sigma + (N_\tau)_\mathbb{R}) / (N_\tau)_\mathbb{R}$, for $\sigma \supseteq \tau$, where N_τ is the integer fan of τ in N. Moreover, we denote by $\Sigma^{(k)}$ the set of all cones in Σ of codimension k.

If $\tau \in \Sigma^{(k+1)}$ is contained in a cone $\sigma \in \Sigma^{(k)}$, let $v_{\sigma/\tau} \in N/N_\tau$ be the primitive generator of the ray $\overline{\sigma}$ in Σ_τ.

Definition 3.1.1. A function $c\colon \Sigma^{(k)} \to \mathbb{Z}$ is called a *Minkowski weight* if it satisfies the *balancing condition*: for every $\tau \in \Sigma^{(k+1)}$, $\sum_{\sigma \supset \tau} c(\sigma) v_{\sigma/\tau} = 0$ in N/N_τ.

Remark 3.1.2. For every fan Σ there are two special Minkowski weights.

- If $\dim \Sigma = k$, the $n - k$ Minkowski weights trivially satisfy the balancing condition.
- If $D = \sum u_\rho D_\rho$ is a Cartier divisor on $\mathbb{P}_\Sigma$, then the Cartier data of D is a Minkowski weight. We can see this via the condition on the intersection product. In fact, for any compactification of $\mathbb{P}_\Sigma$, if a curve C is given by $C = \overline{\mathcal{O}(\tau)}$, with $\tau = \sigma_1 \cap \sigma_2$, the intersection product is

$$(D.C) = \langle m_{\sigma_1} - m_{\sigma_2}, u_1 \rangle = \langle m_{\sigma_2} - m_{\sigma_1}, u_2 \rangle = -\langle m_{\sigma_1} - m_{\sigma_2}, u_2 \rangle.$$

Theorem 3.1.3. [FS97, Main Theorem] *If $\mathbb{P}_\Sigma$ is a smooth complete toric variety, the Chow cohomology $A^{n-k}(\mathbb{P}_\Sigma)$ is canonically isomorphic to the space of codimension $n - k$ Minkowski weights.*

We will now see how tropicalization gives this correspondence.

Every tropical compactification satisfies the combinatorial normal crossing condition with the T-invariant divisors. If $\dim Y = k$, this property naturally induces a $n-k$ Minkowski weight on $\mathrm{Trop}(Y)$.

Now assume that the valuation is trivial, so that $\mathrm{Trop}(Y)$ is a fan Σ. The closure $\overline{Y}$ of Y in $\mathbb{P}_\Sigma$ meets every torus orbit $\mathcal{O}(\tau)$ properly.

Furthermore, we have that $\dim(\mathrm{Supp}(\overline{\mathcal{O}(\sigma)}) \cap \overline{Y}) = 0$ for any facet $\sigma \in \Sigma(k)$ of $\mathrm{Trop}(Y)$. We can thus define a $n-k$ Minkowski weight as the tropical multiplicity

$$m(\sigma) := \deg\left([\overline{Y}] \cdot V(\sigma)\right),$$

where $[\overline{Y}]$ represents the cycle associated with $\overline{Y}$ in $\mathbb{P}_\Sigma$

Definition 3.1.4. Let Y be a subvariety of $\mathbb{P}_\Sigma$ of dimension k. The $n-k$ Minkowski weight c defined above is called the *associated cocycle* of V.

Lemma 3.1.5. [Kat09, Lemma 9.5] *Let Y be a subvariety of $\mathbb{P}_\Sigma$. Suppose Y intersects the torus orbits of $\mathbb{P}_\Sigma$ properly. If c is the associated cocycle of Y, then*

$$c \cap [\mathbb{P}_\Sigma] = [Y] \in A_k(\Sigma).$$

Definition 3.1.6. Let $c \in A^k(\mathbb{P}_\Sigma)$ be a Minkowski weight of codimension k. The *support* of c is

$$\mathrm{Supp}(c) := \bigcup_{\sigma | c(\sigma) \neq 0} \mathrm{cl}(\sigma),$$

the union of the closures of the cones on which c is non-zero.

Remark 3.1.7. $\mathrm{Supp}(c)$ is a union of cones of codimension at least k.

Remark 3.1.8. The support of an associated cocycle of a subvariety Y of $\mathbb{P}_\Sigma$ is $\mathrm{Trop}(Y)$. We put the fan structure on $\mathrm{Supp}(c)$ induced from that of Σ.

Definition 3.1.9. A *mixed Minkowski weight (c, f) of degree k* is a Minkowski weight $c \in A^k(\mathbb{P}_\Sigma)$ together with a continuous function $f\colon \mathrm{Supp}(c) \to \mathbb{R}$ such that the restriction f_σ of f to each cone $\sigma \in \mathrm{Supp}(c)$ is an element of $M/(\sigma^\perp)$.

3.2. **Induced intersection theory and tropicalizations.** Let $\mathbb{P}_\Sigma$ be a smooth n-dimensional toric variety over $\mathbb{C}$.

Let $i\colon Y \to \mathbb{P}_\Sigma$ be a smooth k-dimensional subvariety of $\mathbb{P}_\Sigma$.

Suppose that Y intersects torus orbits properly. Then Y is stratified by its intersection with the strata of $\mathbb{P}_\Sigma$. Let $c \in A^k(\mathbb{P}_\Sigma)$ be the associated cocycle of Y. Then $\mathrm{Supp}(c) = \mathrm{Trop}(Y)$.

Since $c \cap [\mathbb{P}_\Sigma] = [Y]$, for $d \in A^{n-k}(\mathbb{P}_\Sigma)$,

$$i^*d \cap [Y] = d \cap (c \cap [\mathbb{P}_\Sigma]) = (d \cup c) \cap [\mathbb{P}_\Sigma].$$

Definition 3.1. Let $d_1, d_2 \in A^1(\mathbb{P}_\Sigma)$. d_1 and d_2 are *strata-equivalent* on Y if

$\deg((d_1 \cup c) \cap [V(\tau)]) = \deg((d_2 \cup c) \cap [V(\tau)])$ for all $\tau \in \Sigma^{(k-1)}$ with $\tau \in \mathrm{Supp}(c)$.

Lemma 3.2. [Kat12, Lemma 6.2] *If $d_1, d_2 \in A^1(\mathbb{P}_\Sigma)$ are strata-equivalent on Y, the $d_1 = d_2$ in $A^*(\mathbb{P}_\Sigma)/(\ker(\cup c))$.*

Definition 3.3. Let $d \in A^1(\mathbb{P}_\Sigma)$. A piecewise-linear function f on $\mathrm{Trop}(Y)$ *lifts d on Y* if $\deg(i^*d \cap [i^{-1}V(\tau)]) = \kappa(c, f)(\tau)$ for all $\tau \in \Sigma^{(k+1)}, \tau \subset \mathrm{Trop}(Y)$, where $\eta^*\colon A^*_T(\mathbb{P}_\Sigma) \to A^*(\mathbb{P}_\Sigma)$ and $\kappa(c, f)(\tau) = c \cup \eta^* f$.

Theorem 3.4. [Kat12, Theorem 6.4] *Let V be a k-dimensional schön subvariety of a smooth $\mathbb{P}_\Sigma$ and suppose that $(Y, \mathbb{P}_\Sigma)$ is a tropical. Let $d_1, \dots, d_k \in A^1(\mathbb{P}_\Sigma)$ and let $f_1, \dots, f_k$ be piecewise-linear functions on* $\mathrm{Trop}(Y)$ *lifting $d_1, \dots, d_k$. Then*

$$\deg(f_1 \cdot \dots \cdot f_k \cdot \mathrm{Trop}(Y)) = \deg((i^* d_1 \cup \dots \cup i^* d_k) \cap [Y]).$$

4. Tropicalization of line bundles and the toric Zariski–Riemann Space

Building on the results in the previous sections, we start our construction.

Construction 4.1. Let Y be a very affine and irreducible subvariety of an algebraic torus T. By combining Theorem 2.3.1 and Proposition 2.1.3, we can take a tropical compactification $\overline{Y}$ of Y in a smooth toric variety $\mathbb{P}_\Sigma$, whose fan Σ is supported on the non-archimedean amoeba $\mathrm{Trop}(Y)$ of Y.

Let $\dim Y = k$, $\dim \mathbb{P}_\Sigma = n$. We fix our notations in the following diagram.

$$\begin{tikzcd} Y \arrow[r, hook, "i_Y"] \arrow[d, hook, "j_Y"] & \overline{Y} \arrow[d, hook, "j"] \\ T \arrow[r, hook, "i"] & \mathbb{P}_\Sigma \end{tikzcd}$$

Recall that $\mathrm{Trop}(\overline{Y}) = \mathrm{Trop}(Y)$.

Since $(Y, \mathbb{P}_\Sigma)$ is a tropical pair (this is the case, for instance, when we take the trivial valuation), in view of Theorem 2.3.1 and of Proposition 2.3.3, $\overline{Y}$ intersects torus orbits properly and more, $\mathrm{Trop}(\overline{Y})$ is a union of cones of Σ. Moreover, because of the combinatorial normal crossing condition, we have that

$$\dim(\overline{Y} \cap \overline{\mathcal{O}(\tau)}) = \dim(\overline{Y}) + \dim(\overline{\mathcal{O}(\tau)}) - \dim(\mathbb{P}_\Sigma) \text{ for every } \tau \in \Sigma.$$

We will say that a set of $\rho_1, \dots, \rho_{k-1} \in \Sigma(1)$ is admissible if the cone

$$\langle \rho_1, \dots, \rho_{k-1} \rangle = \tau \in \Sigma(k-1).$$

In particular $\dim(\overline{\mathcal{O}(\tau)}) = n - k + 1$. So, because of the trasversality with the orbits, we have that

$$\dim(\overline{Y} \cap \overline{\mathcal{O}(\tau)}) = 1.$$

Now, let $\mathcal{L}$ be a line bundle on $\overline{Y}$. To every cone $\tau \in \Sigma(k-1)$ we will assign a weight in the following way:

$$w(\tau) := \deg\left(\mathcal{L}|_{(D_1 \cdot \dots \cdot D_{k-1})}\right),$$

where $D_i := \overline{\mathcal{O}(\rho_i)}$ for $\rho_1, \dots, \rho_{k-1} \in \Sigma(1)$ an admissible set for τ.

Clearly, once we have assigned weights coming from the line bundle on $\overline{Y}$, we want to be able to describe a tropical line bundle via Minkowski weights.

Definition 4.2. We define the tropicalization of $\mathcal{L} \subseteq \overline{Y}$ as the toric line bundle $D = \sum d_\rho D_\rho \subseteq \mathbb{P}_\Sigma$ characterized by the following condition. For every $Z = \overline{\mathcal{O}(\tau)}$, with $\tau \in \Sigma(k-1)$,

$$\deg(D.[Z])_{\overline{Y}} = w(\tau).$$

We will denote this toric line bundle by $\mathrm{Trop}(\mathcal{L})$.

We need to show that this is a good definition.

Proposition 4.3. *The divisor D in Definition 4.2 is (uniquely) well-defined by $\mathcal{L}$ up to refinement of the underlying fan,* i.e. *up to toric morphisms.*

Proof. The first observation to be made is that if $\overline{\mathcal{O}(\gamma)} \cap \overline{Y} = \emptyset$ then $\gamma \notin \Sigma$.

Let us then consider a cone $\gamma \in \Sigma(s)$ satisfying the combinatorial normal crossing intersection condition with $\overline{Y}$. This implies that

$$Z := \overline{\mathcal{O}(\gamma)} = \bigcap_{i=1}^{s} D_{\rho_i}.$$

Performing the toric blow-up of Z means to add a 1-dimensional ray $\rho_E \in \Sigma(1)$ corresponding to the minimal generator $u_E = \sum u_{\rho_i}$. In particular, denoting by π the associated toric morphism and D_E the T-invariant divisor corresponding to u_E, with abuse of notation we can write

$$\pi^* D = \sum_{\rho \neq \rho_E} d_\rho D_\rho - \langle m, u_E \rangle D_E,$$

where χ^m is a local equation for D in an open U_σ associated with a maximal cone containing $u_E \in \sigma(1)$.

We can thus consider the following diagram

$$\begin{tikzcd}
\pi^{-1}\overline{Y} \arrow[r, hook, "\overline{j}"] \arrow[d, "\pi_Y"'] & \mathbb{P}_{\Sigma'} \arrow[d, "\pi"] \\
\overline{Y} \arrow[r, hook, "j"] & \mathbb{P}_{\Sigma}
\end{tikzcd}$$

and we need to show that $\pi^*(D) = \mathrm{Trop}(\pi^*\mathcal{L})$, *i.e.* that the tropicalization commutes with the pullback of π.

It is simple to check that for every $(k-1)$-dimensional cone τ, $w(\tau) = 0$, since it corresponds to intersecting $\pi^*\mathcal{L}$ with an exceptional curve. We need to show that this conditiond is preserved by the degree of $\pi^* D$.

The only T-invariant schemes we need to consider are those of the form

$$W = D_{\rho_1} \cap \ldots \cap D_{\rho_{k-2}} \cap D_E,$$

where $\{\rho_1, \ldots, \rho_{k-2}\} \in \tau(1)$. Note again that it is enough to consider all the subcones $\tau \in \Sigma(k-1)$.

Let us compute $\deg(\pi^* D.[W])_{\pi^{-1}\overline{Y}}$. We will make use of the fact that the divisor associated with a character is principal. In particular we have that

$$\mathrm{div}(\chi^m) = \sum \langle m, u_\rho \rangle D_\rho \sim 0 \quad \Longrightarrow \quad \langle m, u_E \rangle D_E \sim - \sum_{u_\rho \neq u_E} \langle m, u_\rho \rangle D_\rho.$$

Therefore we get

$$\begin{aligned}\deg(\pi^* D.[W])_{\pi^{-1}\overline{Y}} = \left((\sum_{u_\rho \neq u_E} d_\rho D_\rho + d_E D_E).[D_{\rho_1} \cap \ldots \cap D_{\rho_{k-2}} \cap D_E] \right)_{\pi^{-1}\overline{Y}} = \\ = \left((\sum_{u_\rho \neq u_E} d_\rho D_\rho + \sum_{u_\rho \neq u_E} \langle m, u_\rho \rangle D_\rho).[D_{\rho_1} \cap \ldots \cap D_{\rho_{k-2}} \cap D_E] \right)_{\pi^{-1}\overline{Y}} = \\ = \left((\sum_{u_\rho \neq u_E} (d_\rho + \langle m, u_\rho \rangle) D_\rho).[D_{\rho_1} \cap \ldots \cap D_{\rho_{k-2}} \cap D_E] \right)_{\pi^{-1}\overline{Y}}\end{aligned}$$

and we can conclude, since $d_\rho + \langle m, u_\rho \rangle = 0$ if ρ is in the maximal cone associated with m (and we are done), while if ρ is not in the maximal cone associated with m then the intersection with $[W]_{\pi^{-1}\overline{Y}}$ is zero by dimension count. □

Definition 4.4. We define a *tropical line bundle of* $Y \subseteq T$ as any line bundle on a tropical compactification $\overline{Y}$ of Y.

Theorem 4.5. *The set of tropical line bundles of a closed and irreducible subvariety Y of an algebraic torus T is isomorphic to the set of toric line bundles in some compact toric variety $\mathbb{P}_\Sigma$ such that $(Y, \mathbb{P}_\Sigma)$ is a tropical pair.*

Proof. We have already seen one direction, since to every line bundle on a tropical compactification we associated a toric divisor on the corrisponding toric variety.

For the converse, let us assume that $\overline{Y} \hookrightarrow \mathbb{P}_\Sigma$ is a tropical compactification and let $D = \sum u_\rho D_\rho$ be a T-invariant divisor on $\mathbb{P}_\Sigma$. We claim that there exists a line bundle $\mathcal{L}$ on $\overline{Y}$ such that $\mathrm{Trop}(\mathcal{L})$ is linearly equivalent to D.

Let us try to define $\mathcal{L}$ in the most intuitive way. Let $\overline{D}$ be defined as

$$\overline{D} := \sum d_\rho (D_\rho|_{\overline{Y}}).$$

We already know that, being a tropical compactification, $D_\rho|_{\overline{Y}}$ is a divisor on $\overline{Y}$. It would be enough to show that for every $Z := \overline{\mathcal{O}(\gamma)}$, $\gamma \in \Sigma(s)$, we have that

$$\deg(D.[Z])_{\overline{Y}} = D_{\overline{Y}}.[Z]_{\overline{Y}},$$

and this holds by [Ful98, Chapter 8]. □

4.1. **Toric Rieman–Zariski Space.** The following notation and results are from [BFJ09] and [BdFF12]. In this subsection all varieties are defined over an algebraically closed field k of characteristic 0.

The Riemann–Zariski space of X is the projective limit

$$\mathfrak{X} := \varprojlim_\pi X_\pi$$

where $\pi \colon X_\pi \to X$ is a proper birational morphism.

The group of Weil and Cartier b-divisors over X are respectively defined as

$$\mathrm{Div}(\mathfrak{X}) := \varprojlim_\pi \mathrm{Div}(X_\pi) \quad \text{and}$$

$$\mathrm{CDiv}(\mathfrak{X}) := \varinjlim_\pi \mathrm{CDiv}(X_\pi),$$

where the first limit is taken with respect to the pushforward and the second with respect to the pullback. We will say that a Cartier b-divisor C is *determined* on X_π for a model X_π over X such that $C_{\pi'} = f^*C_\pi$ for every $f\colon X_{\pi'} \to X_\pi$.

Let $\mathfrak{a}$ be a coherent fractional ideal sheaf on X. If $X_\pi \to X$ is the normalized blow-up of X along $\mathfrak{a}$, we will denote $Z(\mathfrak{a})$ the Cartier b-divisor determined by $\mathfrak{a} \cdot \mathcal{O}_{X_\pi}$.

Lemma 4.1.1 ([BdFF12], Lemma 1.8). *A Cartier b-divisor $C \in \mathrm{CDiv}(\mathfrak{X})$ is of the form $Z(\mathfrak{a})$ if and only if C is relatively globally generated over X.*

Definition 4.1.2 ([BdFF12], Definition 2.3). Let D be an $\mathbb{R}$-Weil divisor on X. The *nef envelope* $\mathrm{Env}(D)$ is the $\mathbb{R}$-Weil b-divisor associated with the graded sequence $\mathfrak{a}_\bullet = \{\mathcal{O}_X(mD)\}_{m\geq 1}$, where

$$Z(\mathfrak{a}_\bullet) := \lim_{m\to\infty} \frac{1}{m} Z(\mathfrak{a}_m).$$

Lemma 4.1.3 ([BdFF12], Lemma 2.11 and Corollary 2.13). *We have that*

(1) *$Z(\mathfrak{a}_\bullet)$ is X-nef and*
(2) *$\mathrm{Env}(D)$ is the largest X-nef $\mathbb{R}$-Weil b-divisor W such that $\pi_*(W_\pi) \leq D$.*

From [BFJ09, section 1.5], we can restrict the whole definition to toric blow-ups $\pi\colon X_\pi \to X$, where X is a toric variety, obtaining the *toric Riemann–Zariski space* $\mathfrak{X}_{tor}$ and a toric Néron-Severi space $CN^1(\mathfrak{X}_{tor}) \subseteq CN^1(\mathfrak{X})$. It is simple to check that $CN^1\mathfrak{X}_{tor})$ is isomorphic to the space of functions $g\colon N_\mathbb{R} \to \mathbb{R}$ that are piecewice linear with respect to some rational fan decomposition of $N_\mathbb{R}$ modulo linear forms.

Definition 4.1.4. The toric Riemann–Zariski space induced by the tropicalization of $Y \subseteq T$ is

$$\mathfrak{X}^{\Sigma}_{tor},$$

the toric Riemann–Zariski space associated to $X = \mathbb{P}_\Sigma$ where $\mathrm{supp}(\Sigma) = \mathrm{supp}(\mathrm{Trop}(Y))$ is constructed with respect to the trivial valuation.

Let us try to understand the structure of this object. The toric varieties $\mathbb{P}_\Sigma$ that we are considering are not complete, because in the construction induced by the tropicalization we do omit all the cones corresponding to orbits whose closure does not intersect the compactification.

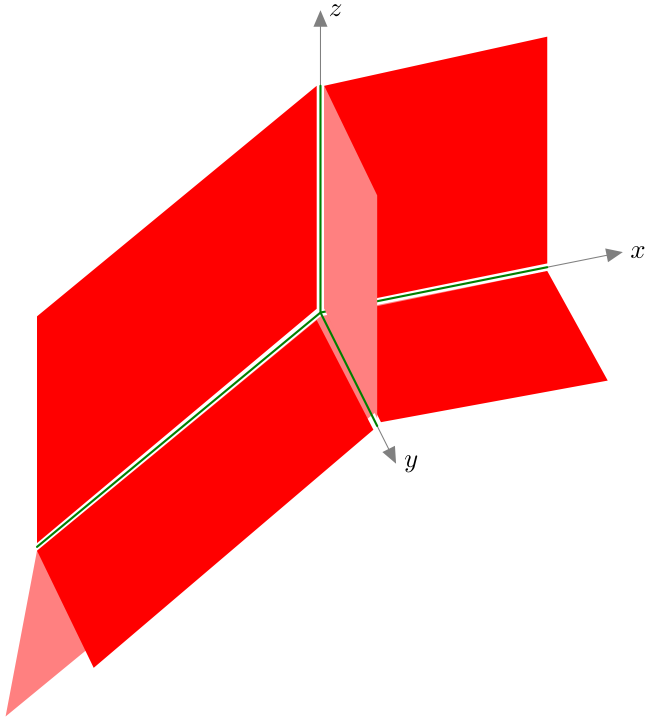
This corresponds to identifying toric bundles isomorphic in codimesion k (depending on the dimesion of the tropical variety). At the level of Zariski–Riemann spaces, this is equivalent to select only the valuations supported in the points at infinity of our initial closed scheme $Y \subseteq T$. We deduce the following.

Proposition 4.1.5. *$\mathfrak{X}^{\Sigma}_{tor}$ is an open subspace of $\mathfrak{X}^{i}_{tor}$, where X^i is any compactification of the smooth variety $\mathbb{P}_\Sigma$ obtained by the tropicalization.*

5. Examples

5.1. **Example 0.** Let us first consider a simple case in $(\mathbb{C}^*)^3$. Let $Y = V(x+y+z-1)$, then the tropical variety will be given by $\mathrm{Sing}(\min\{x,y,z,0\})$, *i.e.* it will be given by the union of the following sections of hyperplanes in $\mathbb{R}^3$:

$$\begin{cases} x=0 \\ 0\leq y,z \end{cases} \quad \begin{cases} y=0 \\ 0\leq x,z \end{cases} \quad \begin{cases} z=0 \\ 0\leq x,y \end{cases} \quad \begin{cases} x=y \\ y\leq 0,z \end{cases} \quad \begin{cases} x=z \\ z\leq 0,y \end{cases} \quad \begin{cases} y=z \\ z\leq 0,x \end{cases}$$



Then the corresponding toric variety $\mathbb{P}_\Sigma$ is

$$\mathbb{P}^3 \backslash \{(1:0:0:0),(0:1:0:0),(0:0:1:0),(0:0:0:1)\}.$$

The easy check is to intersect the closure of the original variety in the toric one and intersect with the $(n-1)$-dimensional orbits.

$$\overline{Y} = V(x+y+z-w) \subseteq \mathbb{P}^3$$

that obviously does not contain the 4 T-invariant points.

In particular, this example doesn't give much information since each divisor only depends on the total degree since the $\rho(\overline{Y}) = 1$ and the construction obviously works.

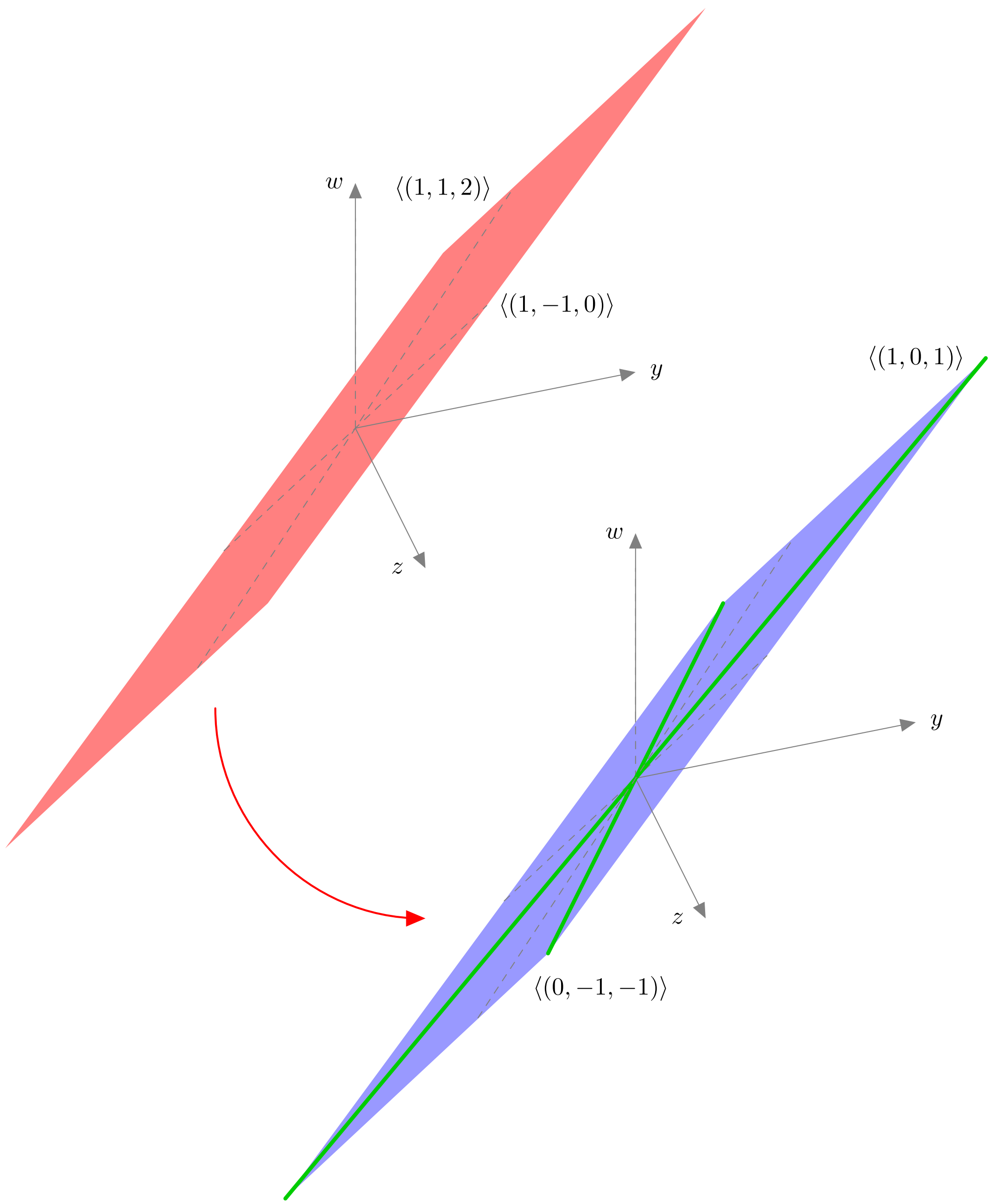
5.2. **Example 1.** We want to describe the example with a ruled surface in $\mathbb{P}^3$. We want to compare the Picard group of the given surface and the one induced by the tropicalization. To start with, let us consider a simple example, the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$.

The local equation for such a variety is given by $Y = V(yz - w)$, so its tropicalization is given by $\mathrm{Trop}(Y) = \{w = x + y\}$. This is a plane in $\mathbb{R}^3$, hence we would have several option for creating a fan supported on the tropicalization. The natural choice is to pick those minimal generators of smallest norm that induce a smooth toric variety. In this case the minimal generators will be $\{(1,0,1),(0,-1,-1),(-1,0,1),(0,1,1)\}$. Considering the local trivialization induced by $\frac{yz}{w} = t$ we obtain the toric variety induced by

$$(y:z) \times (w:x) \times (t) \quad i.e. \quad \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}^*,$$

and also in this case we see that the wanted structure is easily induced.

This will be isomorphic to the *toric Zariski-Riemann space* of $\mathbb{P}^1 \times \mathbb{P}^1$.



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